Relativity

2.1 Velocity Addition Law

If frame B is moving with respect to frame A with velocity v_1, and frame C is moving with respect to frame A with velocity v_2 in the *opposite* direction, then the relative velocity between frames B and C is given by

$$v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Einstein actually derived a more general relation in which frames Band C could move at arbitrary angles with respect to frame A.

An interesting consequence of this relation is that if both v_1 and v_2 are less than c in magnitude, then so is v_3. Thus, one cannot exceed the speed c no matter how one 'adds' up the velocities of several frames. The speed c is the 'speed limit' of nature which no object can ever exceed. Furthermore, if one of the velocities is the speed of light c, then the sum is also c. Thus, the speed of light viewed from any frame is c, which is indeed one of the postulates on which the above equation is based. Finally, note that if v_1 and v_2 are both much smaller than c, then we recover the 'intuitive' law of addition of velocities, $v_3 = v_1 + v_2$.

PROOF:

Suppose, relative to a frame S, a particle has a velocity $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$

where $u_x = dx/dt$ etc. What we require is the velocity of this particle as measured in the frame of reference S' moving with a velocity v_x relative to S. If the particle has coordinate x at time t in S, then the particle will have coordinate x' at time t' in S' where

$x = \gamma(x'+vt')$ and $t = \gamma\,(t'+v_x x'/c^2)$

If the particle is displaced to a new position $x + dx$ at time $t + dt$ in S, then in S' it will be at the position $x' + dx'$ at time $t' + dt'$ where

$$x + dx = \gamma\,(x' + dx' + v_x(t' + dt')$$

$$t + dt = \gamma\,(t' + dt' + v_x\,(x' + dx')/c^2)$$

And hence

$$dx = (dx' + v_x/dt')$$

$$dt = \gamma(dt' + v_x dx'/c^2)$$

So that, $u_x = \frac{dx}{dt} = \frac{dx'+v_x dt'}{dt'+v_x dx'/c^2} = \frac{dx'/dt'+v_x}{1+v_x/c^2(\frac{dx'}{dt'})} = \frac{u_x'+v_x}{1+v_x u_x'/c^2}$

where $u'_x = dx'/dt'$ is the *X* velocity of the particle in the *S′* frame of reference.

2.2 Doppler Shift

If light waves moving in some direction have a frequency ν in a frame A, then they have a frequency,

$$\nu' = \nu\sqrt{\frac{1 - v/c}{1 + v/c}}$$

in a frame which is moving in the *same* direction with velocity v, and a frequency

$$\nu' = \nu\sqrt{\frac{1 + v/c}{1 - v/c}}$$

in a frame which is moving in the *opposite* direction with velocity *v*. Note again that if $v/c \ll 1$, one recovers the more familiar result for Doppler shift $\nu/'\nu = 1 \pm v/c$ (depending on the relative direction of the light waves and the frame of the observer). Using arguments from the classical theory of electromagnetism, Einstein also showed that the energy of the light waves transforms in the same way as the frequency. This is now obvious from the quantum theory of photons; the energy of a photon is proportional to the frequency of the corresponding light wave, the constant of proportionality being given by Planck's constant *h*.

2.3 Mass Energy Relation

Let an object which is at rest in a frame A simultaneously emit two light waves with the same energy *E/2* in opposite directions. Since the two waves carry equal but opposite momenta, the object remains at rest, but its energy decreases by *E*.

By the Doppler shift argument given above, in a frame B which is moving at velocity *v* in one of those directions, the object will appear to lose energy equal to

$$\frac{E}{2}\sqrt{\frac{1 - v/c}{1 + v/c}} + \frac{E}{2}\sqrt{\frac{1 + v/c}{1 - v/c}} = \frac{E}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The difference in energy loss as viewed from the two frames must therefore appear as a difference in kinetic energy seen by frame B. Hence, if *v/c* is very small, in frame B the object loses an amount of kinetic energy given by

$$\frac{E}{\sqrt{1-\frac{v^2}{c^2}}} - E = \frac{1}{2} \times \frac{E}{c^2} \times v^2$$

Since the kinetic energy of an object with mass *M* moving with speed *v* is given by (1/2) Mv^2 (for $v/c \ll 1$), this means that the object has lost an amount of mass given by E/c^2. In other words, a loss in energy of E is equivalent to a loss in mass of E/c^2. This implies an equivalence between the mass and energy content of any object. So, the mass energy relation is $E = mc^2$.